VINNITSA NATIONAL AGRARIAN UNIVERSITY

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LAPLACE TRANSFORM. OPERATOR SUBSTITUTION SCHEMES

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Laplace transform. Operator substitution schemes

THE CONCEPT OF LAPLACE TRANSFORM

DEFINITION

- The Laplace transform is a linear operator that switched a function f(t) to F(s).
- Specifically: $$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

  where: $$s = \sigma + i\omega.$$
- Go from time argument with real input to a complex angular frequency input which is complex.

RESTRICTIONS

- There are two governing factors that determine whether Laplace transforms can be used:
  - f(t) must be at least piecewise continuous for $t \geq 0$
  - $|f(t)| \leq Me^{\gamma t}$ where M and γ are constants

CONTINUITY

- Since the general form of the Laplace transform is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

  it makes sense that f(t) must be at least piecewise continuous for $t \geq 0$.

- If f(t) were very nasty, the integral would not be computable.

LAPLACE TRANSFORM THEORY

- General Theory

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)dt$$

- Example

$$f(t) \equiv 1$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} 1dt = \lim_{\tau\to\infty} \left(\frac{e^{-st}}{-s} \bigg|_0^\tau \right)$$

$$= \lim_{\tau\to\infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) = \frac{1}{s}$$

- Convergence

$$f(t) \equiv e^{t^2}$$

$$\mathcal{L}(f(t)) = \lim_{\tau\to\infty} \int_0^\tau e^{-st} e^{t^2} dt = \lim_{\tau\to\infty} \int_0^\tau e^{t^2 - st} dt = \infty$$

LAPLACE TRANSFORMS

- Some Laplace Transforms
- Wide variety of function can be transformed

- Inverse Transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

- Often requires partial fractions or other manipulation to find a form that is easy to apply the inverse

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^nf(t)$	$F^{(n)}(s)$

RESISTOR

- iv-relation in the time domain

$$v(t) = R \cdot i(t).$$

- By operational Laplace transform:

$$L\{v(t)\} = L\{R \cdot i(t)\} = R \cdot L\{i(t)\} \ ,$$

$$\Rightarrow V(s) = R \cdot I(s).$$

- Physical units: $V(s)$ in volt-seconds, $I(s)$ in ampere-seconds.

INDUCTOR

- *iv*-relation in the time domain

$$v(t) = L \cdot \frac{d}{dt} i(t).$$

- By operational Laplace transform:

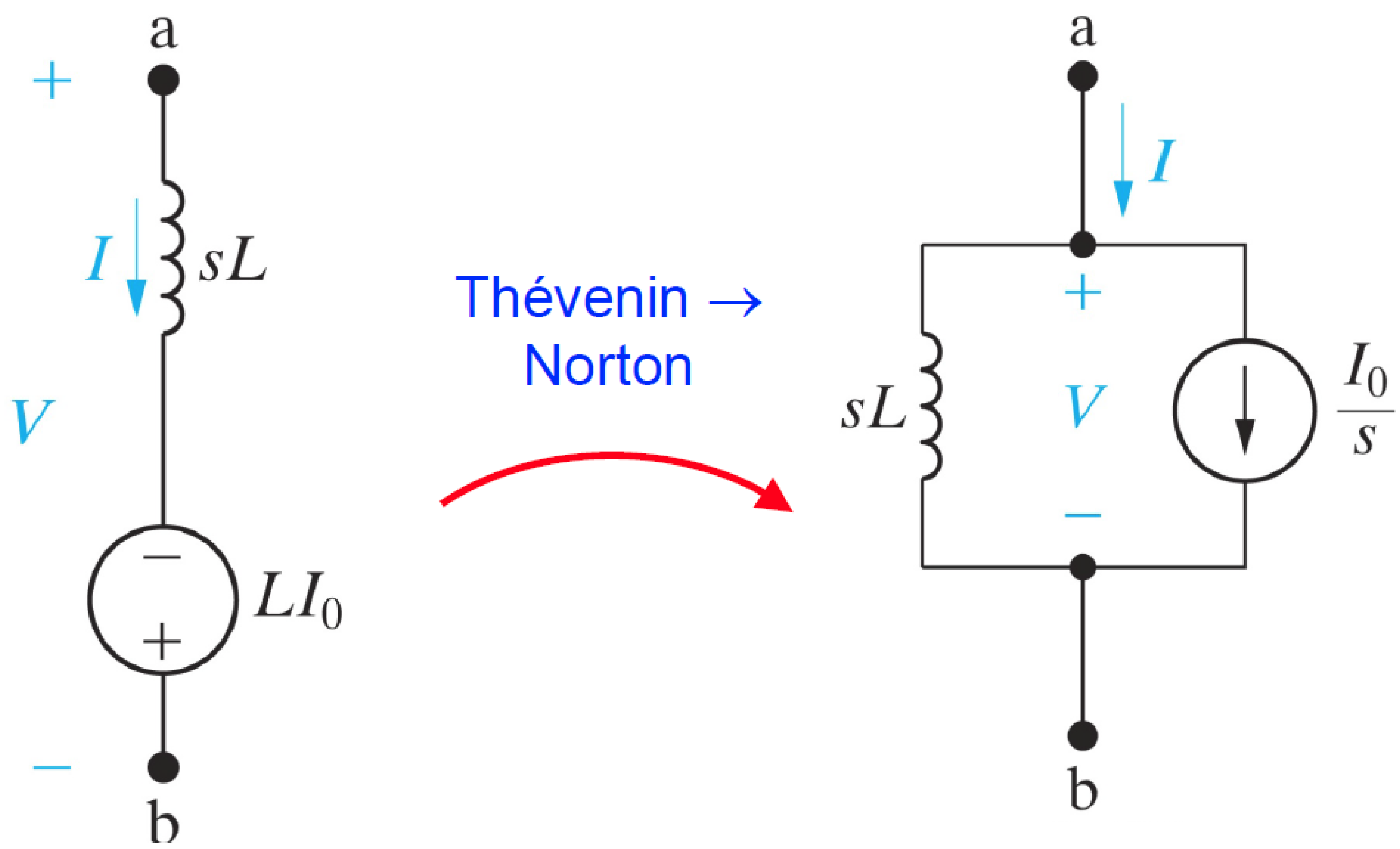
$$L\{v(t)\} = L\{L \cdot i'(t)\} = L \cdot L\{i'(t)\} \ ,$$

$$\Rightarrow V(s) = L \cdot [sI(s) - I_0] = sL \cdot I(s) - LI_0 .$$

initial current

EQUIVALENT CIRCUIT OF AN INDUCTOR

- Series equivalent:
- Parallel equivalent:



CAPACITOR

- *iv*-relation in the time domain

$$i(t) = C \cdot \frac{d}{dt} v(t).$$

- By operational Laplace transform:

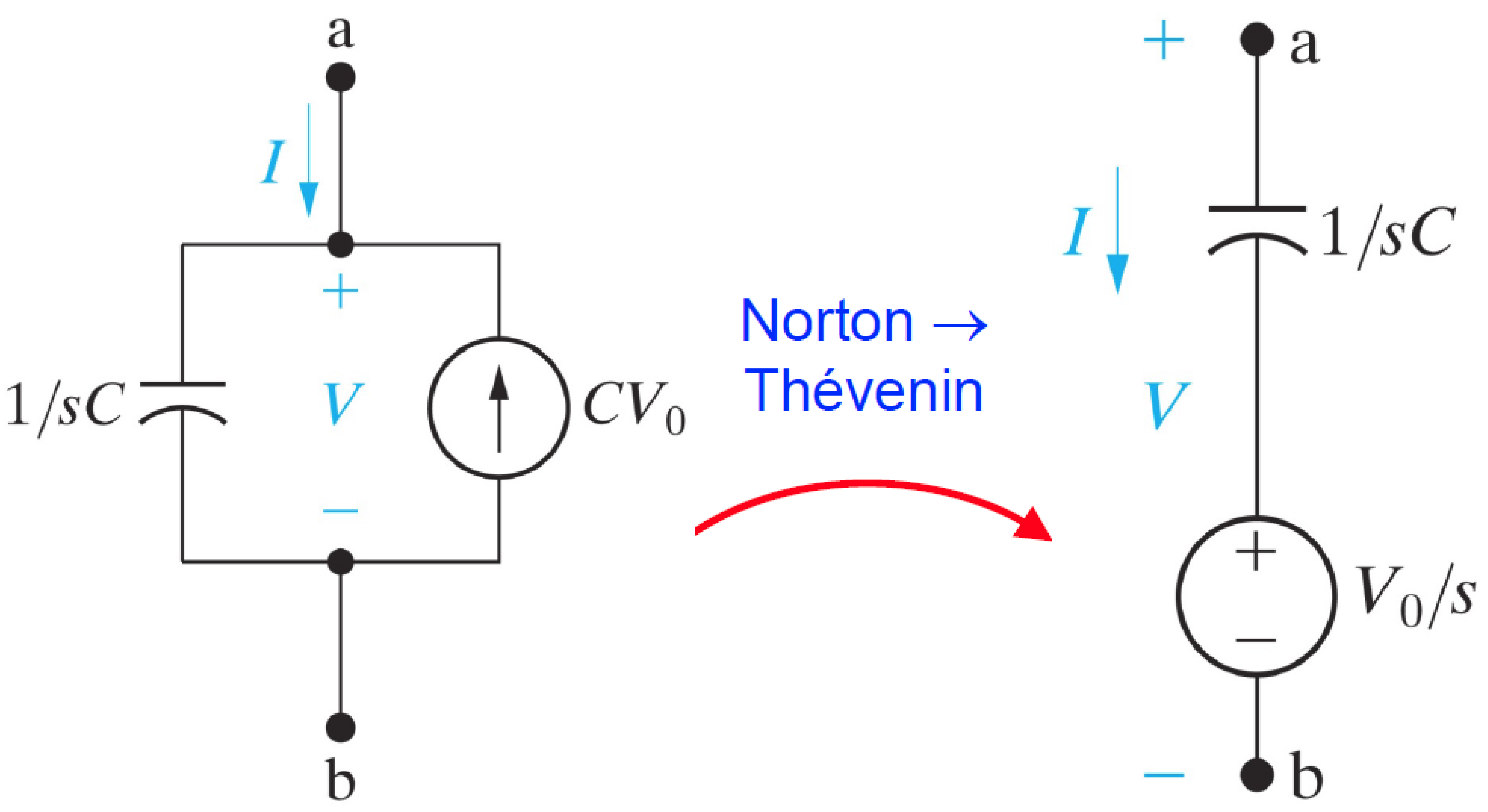
$$L\{i(t)\} = L\{C \cdot v'(t)\} = C \cdot L\{v'(t)\} \ ,$$

$$\Rightarrow I(s) = C \cdot [sV(s) - V_0] = sC \cdot V(s) - CV_0.$$

initial voltage

EQUIVALENT CIRCUIT OF A CAPACITOR

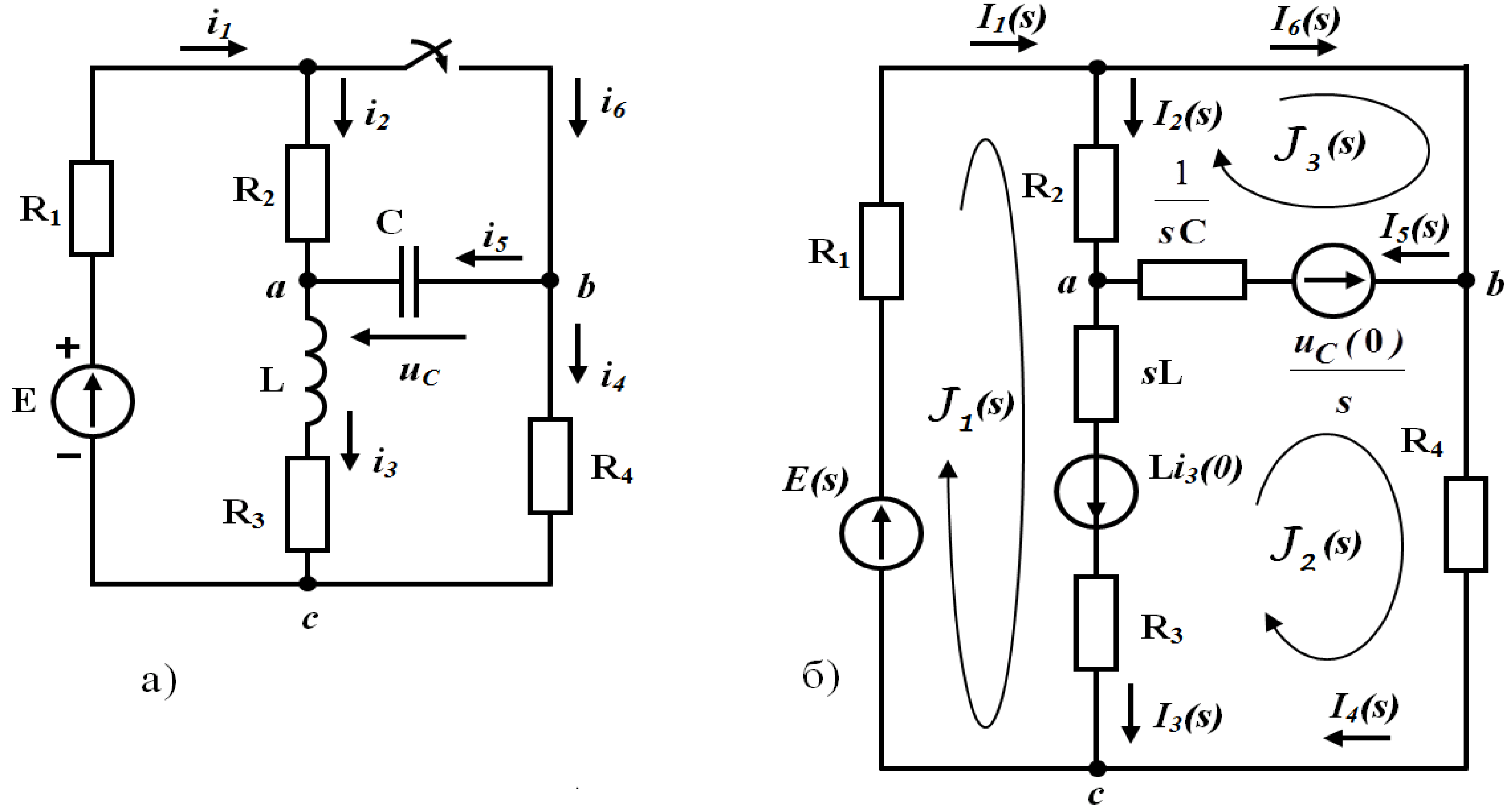
- Parallel equivalent:
- Series equivalent:



AN EXAMPLE OF CONSTRUCTING AN OPERATOR SUBSTITUTION SCHEME

Original scheme

Substitution scheme



INDEPENDENT WORK

Complete an operator substitution scheme for the proposed circle